Clamped Splines

Example

Construct a clamped spline S that passes through the points $(1, 2)$, $(2, 3)$, and $(3, 5)$. $s'(1) = 2$ and $s'(3) = 1$.

Solution

$$s_0(x) = a_0 + b_0(x - 1) + c_0(x - 1)^2 + d_0(x - 1)^3$$

$$s_1(x) = a_1 + b_1(x - 2) + c_1(x - 2)^2 + d_1(x - 2)^3$$

$$2 = f(1) = a_0$$

$$3 = f(2) = a_0 + b_0 + c_0 + d_0$$

$$3 = f(2) = a_1$$

$$5 = f(3) = a_1 + b_1 + c_1 + d_1$$

$$s_0'(2) = s_1'(2): \quad b_0 + 2c_0 + 3d_0 = b_1$$

$$s_0''(2) = s_1''(2): \quad 2c_0 + 6d_0 = 2c_1$$

$$s_0'(1) = 2: \quad b_0 = 2$$

$$s_1'(3) = 1: \quad b_1 + 2c_1 + 3d_1 = 1$$

Solving this system of equations gives the spline as

$$s(x) = \begin{cases} 2 + 2(x-1) - \frac{5}{2}(x-1)^2 + \frac{3}{2}(x-1)^3, & \text{for } x \in [1,2] \\ 3 + \frac{3}{2}(x-2) + 2(x-2)^2 - \frac{3}{2}(x-2)^3, & \text{for } x \in [2,3] \end{cases}$$

Theorem

If f is defined at $a = x_0 < x_1 < \cdots < x_n = b$ and differentiable at a and b, then f has a unique clamped spline interpolant S on the nodes $x_0, x_1, \ldots, x_n$; that is, a spline interpolant that satisfies the clamped boundary conditions $S'(a) = f'(a)$ and $S'(b) = f'(b)$. ■

Proof

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \qquad j = 0, 1, \ldots, n-1$$

Remember Equation (3.20),

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \tag{3.20}$$

Since $f'(a) = S'(a) = S'(x_0) = b_0$, Eq. (3.20) with $j = 0$ implies

$$f'(a) = \frac{1}{h_0}(a_1 - a_0) - \frac{h_0}{3}(2c_0 + c_1)$$

Consequently,

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

Also remember that,

$$b_{j+1} = b_j + h_j(c_j + c_{j+1}). \tag{3.19}$$

On the other hand, $f'(b) = b_n$. So,

$$f'(b) = b_n = b_{n-1} + h_{n-1}(c_{n-1} + c_n)$$

Combination of the above equation and Eq. (3.20) with j=n-1 results in,

$$\begin{aligned} f'(b) &= \frac{a_n - a_{n-1}}{h_{n-1}} - \frac{h_{n-1}}{3}(2c_{n-1} + c_n) + h_{n-1}(c_{n-1} + c_n) \\ &= \frac{a_n - a_{n-1}}{h_{n-1}} + \frac{h_{n-1}}{3}(c_{n-1} + 2c_n), \end{aligned}$$

Consequently,

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

Remember equations (3.21),

$$h_{j-1}c_{j-1} + 2(h_{j-1} + h_j)c_j + h_jc_{j+1} = \frac{3}{h_j}(a_{j+1} - a_j) - \frac{3}{h_{j-1}}(a_j - a_{j-1}), \qquad (3.21)$$

Equations (3.21) together with the equations,

$$2h_0c_0 + h_0c_1 = \frac{3}{h_0}(a_1 - a_0) - 3f'(a)$$

$$h_{n-1}c_{n-1} + 2h_{n-1}c_n = 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1})$$

determine the linear system $\mathbf{A}\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 2h_0 & h_0 & 0 & \cdots & \cdots & \cdots & 0 \\ h_0 & 2(h_0+h_1) & h_1 & \ddots & & & \vdots \\ 0 & h_1 & 2(h_1+h_2) & h_2 & \ddots & & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ & & & & h_{n-2} & 2(h_{n-2}+h_{n-1}) & h_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 & h_{n-1} & 2h_{n-1} \end{bmatrix},$$

$$\mathbf{b} = \begin{bmatrix} \frac{3}{h_0}(a_1 - a_0) - 3f'(a) \\ \frac{3}{h_1}(a_2 - a_1) - \frac{3}{h_0}(a_1 - a_0) \\ \vdots \\ \frac{3}{h_{n-1}}(a_n - a_{n-1}) - \frac{3}{h_{n-2}}(a_{n-1} - a_{n-2}) \\ 3f'(b) - \frac{3}{h_{n-1}}(a_n - a_{n-1}) \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ \vdots \\ c_n \end{bmatrix}.$$

Matrix A is strictly diagonally dominant. A linear system with a matrix of this form has a unique Solution.

Example

Use the data points $(0, 1)$, $(1, e)$, $(2, e^2)$, and $(3, e^3)$ to form a clamped spline $S(x)$ that approximates $f(x) = e^x$. $f'(0) = 1$ and $f'(3) = e^3$.

Solution

$$S_j(x) = a_j + b_j(x - x_j) + c_j(x - x_j)^2 + d_j(x - x_j)^3 \qquad j = 0, 1, \ldots, n - 1$$

$n = 3$, $h_0 = h_1 = h_2 = 1$, $a_0 = 0$, $a_1 = e$, $a_2 = e^2$, and $a_3 = e^3$.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 \\ 1 & 4 & 1 & 0 \\ 0 & 1 & 4 & 1 \\ 0 & 0 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 3(e - 2) \\ 3(e^2 - 2e + 1) \\ 3(e^3 - 2e^2 + e) \\ 3e^2 \end{bmatrix}, \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \end{bmatrix}$$

$A\mathbf{x} = \mathbf{b}$ is equivalent to the system of equations

$$2c_0 + c_1 = 3(e-2),$$

$$c_0 + 4c_1 + c_2 = 3(e^2 - 2e + 1),$$

$$c_1 + 4c_2 + c_3 = 3(e^3 - 2e^2 + e),$$

$$c_2 + 2c_3 = 3e^2.$$

Solving this system simultaneously for c_0, c_1, c_2 and c_3 gives,

$$c_0 = \frac{1}{15}(2e^3 - 12e^2 + 42e - 59) = 0.44468,$$

$$c_1 = \frac{1}{15}(-4e^3 + 24e^2 - 39e + 28) = 1.26548,$$

$$c_2 = \frac{1}{15}(14e^3 - 39e^2 + 24e - 8) = 3.35087,$$

$$c_3 = \frac{1}{15}(-7e^3 + 42e^2 - 12e + 4) = 9.40815.$$

$$b_j = \frac{1}{h_j}(a_{j+1} - a_j) - \frac{h_j}{3}(2c_j + c_{j+1}), \tag{3.20}$$

So,

$$b_0 = 1.00000, \quad b_1 = 2.71016, \quad b_2 = 7.32652$$

Also,

$$c_{j+1} = c_j + 3d_jh_j. \tag{3.17}$$

So,

$$d_0 = 0.27360, \quad d_1 = 0.69513, \quad d_2 = 2.01909$$

Finally,

$$
s(x) = \begin{cases} 1 + x + 0.44468x^2 + 0.27360x^3, & \text{if } 0 \le x < 1, \\ 2.71828 + 2.71016(x-1) + 1.26548(x-1)^2 + 0.69513(x-1)^3, & \text{if } 1 \le x < 2, \\ 7.38906 + 7.32652(x-2) + 3.35087(x-2)^2 + 2.01909(x-2)^3, & \text{if } 2 \le x \le 3. \end{cases}
$$

$$
\int_0^3 e^x \, dx = e^3 - 1 \approx 20.08554 - 1 = 19.08554.
$$

$$
\int_0^3 s(x) \, dx = 19.05965
$$

The absolute error in the integral approximation are

Natural : $|19.08554 - 19.55229| = 0.46675$

Clamped : $|19.08554 - 19.05965| = 0.02589$

HOMEWORK 4:

Exercise Set 3.5: 13,29

Direct Methods for Solving Linear Systems

Direct techniques are methods that theoretically give the exact solution to the system in a finite number of steps.

Linear Systems of Equations

$$
\begin{aligned}
E_1: &\quad a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1,\\
E_2: &\quad a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2,\\
&\quad \vdots\\
E_n: &\quad a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n.
\end{aligned} \tag{6.1}
$$

We use three operations to simplify the linear system given in (6.1):

1. Equation E_i can be multiplied by any nonzero constant λ with the resulting equation used in place of E_i. This operation is denoted $(\lambda E_i) \to (E_i)$.

2. Equation E_j can be multiplied by any constant λ and added to equation E_i with the resulting equation used in place of E_i. This operation is denoted $(E_i + \lambda E_j) \to (E_i)$.

3. Equations E_i and E_j can be transposed in order. This operation is denoted $(E_i) \leftrightarrow (E_j)$.

Illustration

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \phantom{{}- x_3} + 3x_4 = 4, \\
E_2 &: \quad 2x_1 + x_2 - x_3 + x_4 = 1, \\
E_3 &: \quad 3x_1 - x_2 - x_3 + 2x_4 = -3, \\
E_4 &: \quad -x_1 + 2x_2 + 3x_3 - x_4 = 4,
\end{aligned}
$$

By performing:

$(E_2 - 2E_1) \to (E_2)$ $(E_3 - 3E_1) \to (E_3)$ $(E_4 + E_1) \to (E_4)$

We have,

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 + 3x_4 = 4, \\
E_2 &: \quad - x_2 - x_3 - 5x_4 = -7, \\
E_3 &: \quad - 4x_2 - x_3 - 7x_4 = -15, \\
E_4 &: \quad 3x_2 + 3x_3 + 2x_4 = 8.
\end{aligned}
$$

In the new system performing:

$(E_3 - 4E_2) \to (E_3)$ $(E_4 + 3E_2) \to (E_4)$

resuts in:

$$
\begin{aligned}
E_1 &: \quad x_1 + x_2 \qquad\qquad + \ 3x_4 = 4, \\
E_2 &: \qquad - x_2 - x_3 - 5x_4 = -7, \\
E_3 &: \qquad\qquad 3x_3 + 13x_4 = 13, \\
E_4 &: \qquad\qquad\quad - 13x_4 = -13.
\end{aligned}
$$

triangular (or **reduced**) **form**

Using **backward-substitution process**:

$$x_4 = 1$$

$$x_3 = \frac{1}{3}(13 - 13x_4) = \frac{1}{3}(13 - 13) = 0$$

$$x_2 = -(-7 + 5x_4 + x_3) = -(-7 + 5 + 0) = 2$$

$$x_1 = 4 - 3x_4 - x_2 = 4 - 3 - 2 = -1$$

The procedure is called **Gaussian elimination with backward substitution.**

$$
\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\
\vdots \qquad\qquad\qquad & \ \ \vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n,
\end{aligned}
$$

$$\mathbf{Ax} = \mathbf{b}$$

$$
A = [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}
$$

augmented matrix

$$
[A, \mathbf{b}] = \left[\begin{array}{cccc:c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} & b_n \end{array}\right],
$$